Introductory notes on social dynamics

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I. INTRO: WHY WE DEAL WITH SOCIAL PHENOMENA

- Not only in the physical/biological domain, but also in the social domain there are many regularities emerging out of apparently fully erratic behavior: patterns in price fluctuations in finance, network structure in the internet, regular features in traffic, etc.... As the regularities in physics, also those in social context call for interpretation in terms of "microscopic" models. Self-organization is the key point. Internet is as it is not because some engineer designed it, but because millions of independent agents shaped it in that way.

- The discovery of quantitative laws in the collective properties of a large number of people, as revealed, for example, by birth and death rates or crime statistics, was one of the catalysts in the development of statistics, and it led many scientists and philosophers to call for some quantitative understanding of how such precise regularities arise out of the apparently erratic behavior of single individuals. Hobbes, Laplace, Quetelet, Comte, Stuart Mill, and many others shared, to a different extent, this line of thought (Ball). The first formulation of Brownian Motion, due to Bachelier, was actually a theory for the fluctuations of stock prices. His thesis title was "Theory of speculation". Majorana's tenth article is about the application of statistical physics to social systems.

- There is a long tradition of microscopic modeling of social behavior: economics (or microeconomics) is it. However traditional (neoclassical) economics is more a branch of mathematics than something akin to physics. It is not based on observation, rather on assumptions that make analytical computations possible: agents behave rationally, they are perfect calculators, they have complete information. There are lots of anecdotes on how much traditional economists do not believe their own assumptions.

- Agents are not rational, they are adaptive, imitating. See the book "The Social Atom", by M. Buchanan [1], for examples. This calls for different ways to approach the problem.

- Another element that is revolutionizing the field is the huge amount of data available: enormous databases collect data about many human/social activities (election results, phone calls, marketing data) and computers make the analysis of these data much easier. Entire new social phenomena started in the past few decades: internet, electronic financial markets, for example.

- Social scientists are responding to these challenges by progressively including in their modeling elements that are more realistic: imitation, adaptation, noise: they are switching from differential equations to simulations: agent-based modeling. This is the type of models statistical physicists know quite well.

- There is a huge conceptual difference between this emerging field and traditional statistical physics. In usual applications of statistical physics, elementary objects, atoms and molecules, are simple entities, whose behavior is very well known. Complexity arises because of nontrivial collective phenomena, not because of complex behavior of individual entities.

  Humans are the opposite. Their individual behavior and interactions are the complex outcome of many psychological and physiological processes, that are largely unknown. Even if they would be known, they are enormously more complicated than interactions among atoms. It would be impossible to describe them with simple laws and few parameters.

  Any modeling of social agents inevitably involves a huge and unwarranted simplification of the real problem. It is then clear that any investigation of models of social dynamics involves two levels of difficulty. The first is in

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the very definition of sensible and realistic microscopic models; the second is the usual problem of inferring the macroscopic phenomenology out of the microscopic dynamics of such models. Obtaining useful results out of these models may seem a hopeless task.

– One possible way to answer to this observation is to say: yes, the model we consider are totally unrealistic, but we are interested in them because they can provide new challenging and interesting "physics-like" problems.
– Otherwise, one can interpret these models of social dynamics as null models: they allow to single out ingredients of the social behavior and study their effect. This provides information, at a qualitative level, on what is the effect of the single ingredient, providing hints on what is missing.
– But in this respect, statistical physics may bring an important added value, justifying in this way the minimalistic approach. In most situations qualitative (and even some quantitative) properties of large scale phenomena do not depend on the microscopic details of the process. Only higher level features, as symmetries, dimensionality or conservation laws, are relevant for the global behavior. With this concept of *universality* in mind one can then approach the modelization of social systems, trying to include only the simplest and most important properties of single individuals and looking for qualitative features exhibited by models.

In any case, physicists are used to deal with the issue of robustness of their results with respect to the variation of parameters, including system size. This is something that does not belong to the tradition of social sciences.

II. OPINION DYNAMICS

Opinion dynamics deals with a simple conceptual framework. Starting from a disordered initial state, where individuals have different (random) opinions, a dynamics takes place that tends to generate consensus through interactions. The relevant questions one is interested to answer are:

- Is consensus reached?
- What type of consensus?
- How much time does it take?

Essentially we are interested in studying an ordering process, taking place over time. In traditional statistical physics the paradigmatic example of such process is the ordering dynamics of the Ising model [2], which can be seen as a simple model for binary opinion dynamics, with agents influenced by the state of the majority of their interacting partners.

A. Ordering dynamics of Ising model

Consider a collection of N spins (agents) s_i that can assume two values ± 1. Each spin is energetically pushed to be aligned with its nearest neighbors. The total energy is

$$H = -\frac{1}{2} \sum_{<i,j>} s_i s_j, \tag{1}$$

where the sum runs on the pairs of nearest-neighbors spins. Among the possible types of dynamics, the most common (Glauber) takes as elementary move a single spin flip that is accepted with probability

$$P(s_i \rightarrow -s_i) = \frac{1}{1 + \exp[\Delta E/(k_B T)]} \tag{2}$$

where ΔE is the change in energy and T is the temperature. For $T = 0$ the spin flip is accepted with probability 1 if energy is decreased, 0 if energy is increased and 1/2 if the energy is unchanged. Ferromagnetic interactions in Eq. (1) drive the system towards one the two possible ordered states, with all positive or all negative spins. At the same time thermal noise injects fluctuations that tend to destroy order. For low temperature T the ordering tendency wins and long-range order is established in the system, while above a critical temperature T_c the system remains

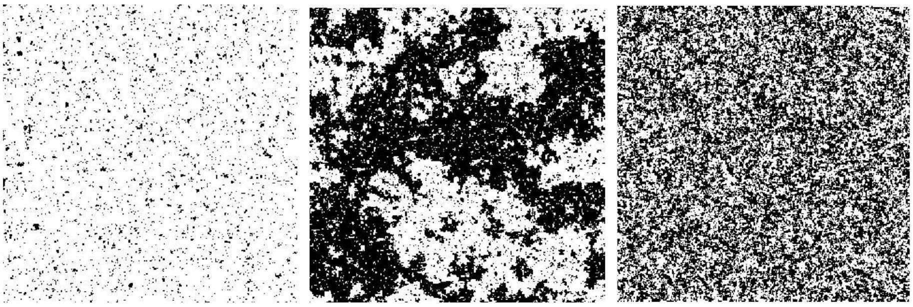

FIG. 1: Snapshots of equilibrium configurations of the Ising model (from left to right) below, at and above T_c.

macroscopically disordered. The transition point is characterized by the average magnetization $m = 1/N \sum_i \langle s_i \rangle$ passing from 0 for $T > T_c$ to a value $m(T) > 0$ for $T < T_c$. The brackets denote the average over different realizations of the dynamics. It is important to stress that above T_c no infinite-range order is established, but on short spatial scales spins are correlated: there are domains of +1 spins (and others of −1 spins) extended over regions of finite size. Below T_c instead these ordered regions extend to infinity (they span the whole system), although at finite temperature some disordered fluctuations are present on short scales (Fig. 1). Not only the equilibrium properties just described, that are attained in the long run, are interesting. A much investigated and nontrivial issue is the way the final ordered state at $T < T_c$ is reached, when the system is initially prepared in a fully disordered state. This ordering dynamics is a prototype for the analogous processes occurring in many models of social dynamics. On short time scales, coexisting ordered domains of small size (both positive and negative) are formed. The subsequent evolution occurs through a *coarsening* process of such domains, which grow larger and larger while their global statistical features remain unchanged over time. This is the dynamic scaling phenomenon: the morphology remains statistically the same if rescaled by the typical domain size, which is the only relevant length in the system and grows over time as a power-law.

An applet with the evolution of Glauber-Ising dynamics can be found at `http://webphysics.davidson.edu/applets/ising/default.html`.

Macroscopically, the dynamic driving force towards order is surface tension. Interfaces between domains of opposite magnetization cost in terms of energy and their contribution can be minimized by making them as straight as possible. This type of ordering is often referred to as curvature-driven and occurs in many of the social systems. The presence of surface tension is a consequence of the tendency of each spin to become aligned with the majority of its neighbors. When the majority does not play a role, the qualitative features of the ordering process change.

The dynamic aspect of the study of social models requires the monitoring of suitable quantities, able to properly identify the buildup of order. The magnetization of the system is not one of such suitable quantities. It is not sensitive to the size of single ordered domains, while it measures their cumulative extension, which is more or less the same during most of the evolution. The appropriate quantity to monitor the ordering process is the correlation function between pairs of spins at distance r from each other,

$$C(r,t) = \langle s_i(t) s_{i+r}(t) \rangle - \langle s_i(t) \rangle^2, \tag{3}$$

where brackets denote averaging over dynamic realizations and an additional average over i is implicit. The temporal variable t is measured as the average number of attempted updates per spin. The dynamic scaling property implies that $C(r,t)$ is a function only of the ratio between the distance and the typical domain size $L(t)$:

$$C(r,t) = F[r/L(t)]. \tag{4}$$

$L(t)$ grows in time as a power-law $t^{1/z}$. The dynamic exponent z is universal, independent of microscopic details, possibly depending only on qualitative features as conservation of the magnetization or space dimensionality. In the Glauber case $z = 2$ in any dimension. Another quantity often used is the density of interfaces $n_a(t) = N_a(t)/N_p$, where N_p is the total number of nearest neighbor pairs and N_a the number of such pairs where the two neighbors are in different states: $n_a = 1/2$ means that disorder is complete, while $n_a = 0$ indicates full consensus.

Dynamics at $T = 0$. Consensus time. Consider now zero-temperature dynamics $T = 0$. One important quantity is the consensus time $T(x,N)$, the time needed for a system of size N and initial density x of up spins to reach consensus. Another quantity is the exit probability $E(x)$, the probability that a system with initial density x of up spins end up in a +1 consensus state.

For the Ising-Glauber dynamics, in all dimensions $d > 1$, the exit probability is (in the thermodynamic limit) simply $E(x) = \theta(x - 1/2)$: the initial majority invades the whole system. The consensus time grows with N as $T \sim N^2$. One

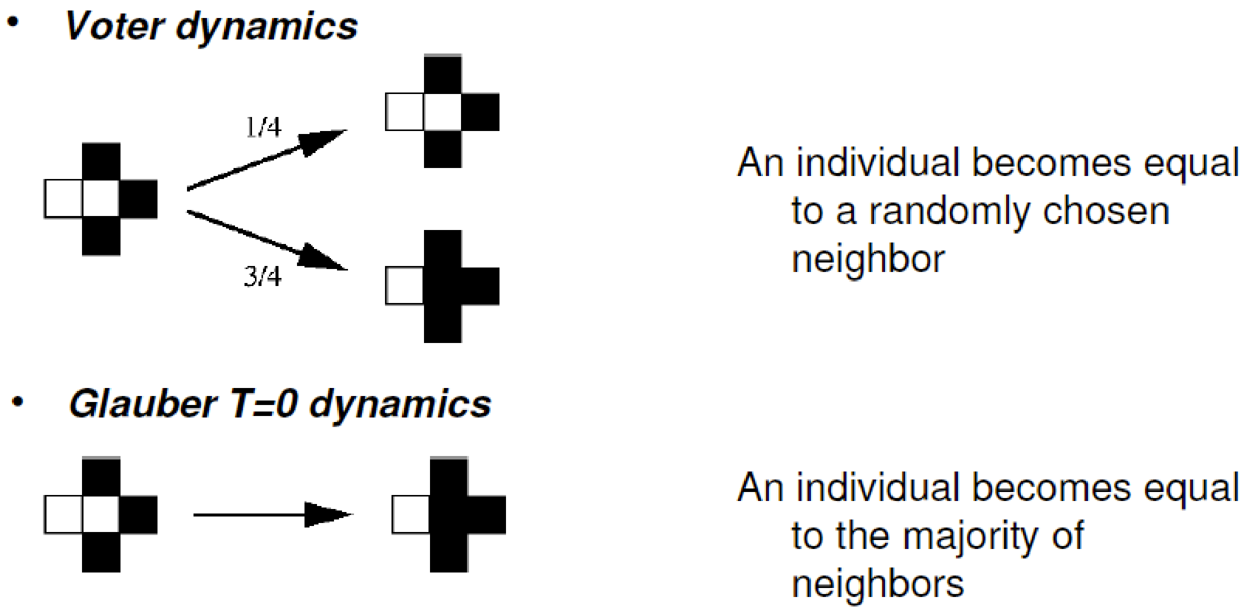

FIG. 2: Definition of voter model compared to Glauber-Ising dynamics.

can see it by simply considering that the system becomes ordered when the size of domains becomes equal to the size of the system

$$L(T) \sim N^{1/d} \tag{5}$$

Since $L(t) = t^{1/2}$ one has

$$T(N) \sim N^{2/d} \tag{6}$$

B. The voter model

Let us consider

- Binary opinions
- Initial fully disordered state
- Short range ferromagnetic interactions

We have already seen a physical realization of this system (Glauber zero-temperature dynamics). But in an interdisciplinary context other types of dynamics have been devised. We consider the voter model. At each time step a site is selected at random and it copies the state of randomly chosen nearest neighbor (Fig. 2)

This model appears in several different contexts

- Coalescing random walks (backward in time).
- Dimer-dimer surface reactions
- Neutral models for plant populations (Hubbell). Plants of two species with equal mortality rate, equal fecundity and nearest-neighbor dispersal.

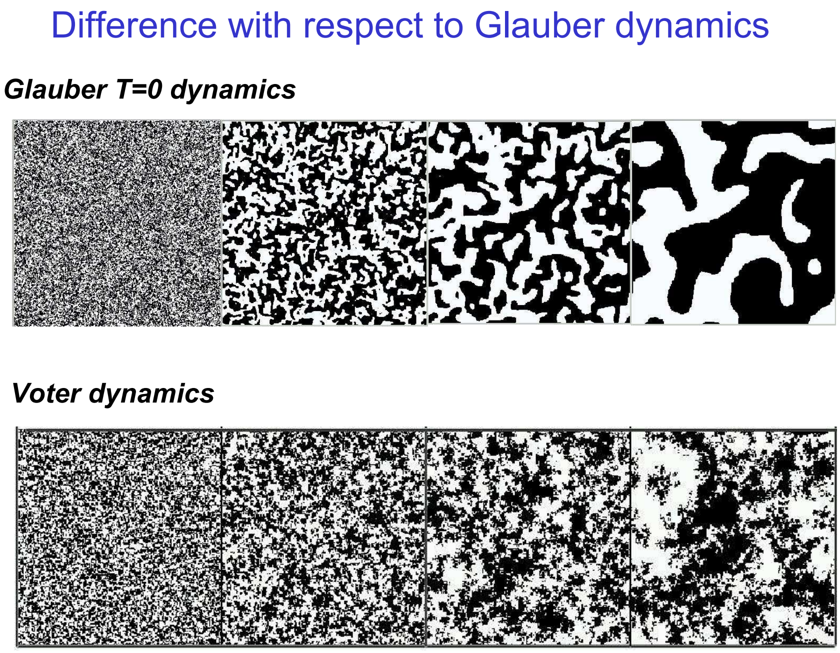

FIG. 3: Comparison of the evolution between Glauber and voter

One dimension In one dimension the dynamics of the voter model can be interpreted in terms of the dynamics of boundaries between domains of up and down spins. The elementary dynamical step implies a diffusive motion of a domain boundary. When two domains meet, they coalesce and cannot be reformed. Hence one can map the dynamics onto the reaction-diffusion system

$$A + A \to 0 \tag{7}$$

which has been studied at length in the literature. This mapping allows immediately the qualitative determination of the consensus time.

A simple nonrigorous argument for the scaling of domain walls: $n(t)$ is the density of walls. $l \sim 1/n$ is the typical domain length. The typical time for the collision between two walls is $\Delta t \sim l^2/D$. During this time the decrease of the density n is $\Delta n \approx -n$, because typically each wall will collide. Hence

$$\dot{n} = \frac{\Delta n}{\Delta t} \approx -Dn^3 \tag{8}$$

from which

$$n^2 \sim \frac{1}{Dt} \Rightarrow n(t) \sim \frac{1}{\sqrt{Dt}} \tag{9}$$

This implies that the total time to reach consensus scales as

$$T(N) \sim N^2 \tag{10}$$

In one dimension it is easy to check that Glauber zero-temperature dynamics has exactly the same microscopic dynamics.

Two dimensions In two dimensions the two models are actually different. This turns out very clearly by comparing patterns of evolution of the two models (see Fig. 3)

The physical origin of the different behavior is made clear by analyzing the evolution of a droplet of one species immersed in the other

For Glauber the droplet shrinks coherently because there is a positive surface tension that tends to reduce the length of the interface. Cahn-Allen argument describes this phenomenon. In this case the minority phase deterministically tends to disappear.

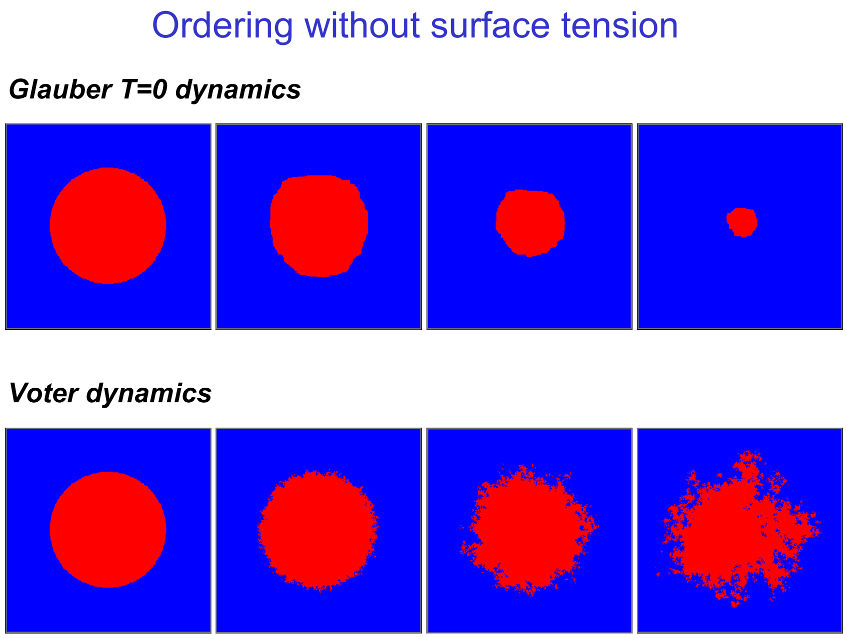

FIG. 4: Comparison of droplet evolution between Glauber and voter

For voter there is no surface tension. The radius of the droplet on average remains constant. There is much more noise, so that the interface is much more corrugated, but on average, the number of minority elements remains constant. Actually at some point the droplet disappears (or conquers the whole system), but this occurs for a stochastic fluctuation which by chance reaches the consensus state. There is no net drift towards consensus.

More than two dimensions For dimensions higher than 2 in the thermodynamic limit the system does not order: it remains forever in a stationary state with rather small domains of the two species dynamically competing. In a finite system, eventually the system reaches consensus, but again this is not due to a coherent tendency toward it, but rather to a stochastic fluctuation.

C. MF solution

We consider a complete graph: each node is connected to all others.

Derivation of Fokker-Planck equation Let $\rho(t)$ be the density of voters in state 1. It completely defines the state of the system. In each update event $\rho(t) \to \rho(t) \pm \delta\rho$, with $\delta\rho = 1/N$, corresponding to the respective state changes $0 \to 1$ or $1 \to 0$. The probabilities for these events are

$$R(\rho) \equiv Prob[\rho \to \rho + \delta\rho] = (1-\rho)\rho \qquad (11)$$

$$L(\rho) \equiv Prob[\rho \to \rho - \delta\rho] = \rho(1-\rho) \qquad (12)$$

$$(13)$$

Let $P(\rho, t)$ the probability density of $+1$ at time t. The Master Equation for it is

$$P(\rho, t+\delta t) = R(\rho - \delta\rho)P(\rho - \delta\rho, t) + L(\rho + \delta\rho)P(\rho + \delta\rho, t) + [1 - R(\rho) - L(\rho)]P(\rho, t) \qquad (14)$$

where $\delta t = 1/N$.

We now expand Eq. (14) to second order in $\delta\rho$ and first in δt.

$$P(\rho \pm \delta\rho, t) = P(\rho, t) \pm \frac{\partial P(\rho, t)}{\partial \rho}\delta\rho + \frac{1}{2}\frac{\partial^2 P(\rho, t)}{\partial \rho^2}(\delta\rho)^2 \qquad (15)$$

$$P(\rho, t+\delta t) = P(\rho, t) + \frac{\partial P(\rho, t)}{\partial t}\delta t \qquad (16)$$

$$R(\rho-\delta\rho)=R(\rho)-\frac{\partial R(\rho)}{\partial\rho}(\delta\rho)+\frac{1}{2}\frac{\partial^2 R(\rho)}{\partial\rho^2}(\delta\rho)^2 \tag{17}$$

$$L(\rho+\delta\rho)=L(\rho)+\frac{\partial L(\rho)}{\partial\rho}(\delta\rho)+\frac{1}{2}\frac{\partial^2 L(\rho)}{\partial\rho^2}(\delta\rho)^2 \tag{18}$$

obtaining (forward Kolmogorov or Fokker-Planck equation)

$$\frac{\partial P(\rho,t)}{\partial t}=-\frac{\partial}{\partial\rho}[v(\rho)P(\rho,t)]+\frac{\partial^2}{\partial\rho^2}[D(\rho)P(\rho,t)] \tag{19}$$

where the drift coefficient is

$$v(\rho)=\frac{\delta\rho}{\delta t}[R(\rho)-L(\rho)] \tag{20}$$

and the diffusion coefficient is

$$D(\rho)=\frac{1}{2}\frac{(\delta\rho)^2}{\delta t}[R(\rho)+L(\rho)] \tag{21}$$

All this is fully general.

In the case of the voter model, the symmetric form of R and L implies $v(\rho)=0$. There is no drift; diffusion drives the system towards one of the two ordered absorbing systems. Hence the Fokker-Planck equation becomes

$$\frac{\partial P(\rho,t)}{\partial t}=\frac{1}{N}\frac{\partial^2}{\partial\rho^2}[\rho(1-\rho)P(\rho,t)] \tag{22}$$

Exit probability In a similar fashion the equation for the exit probability with initial density ρ is:

$$\epsilon(\rho)=R(\rho)\epsilon(\rho+\delta\rho)+L(\rho)\epsilon(\rho-\delta\rho)+[1-R(\rho)-L(\rho)]\epsilon(\rho) \tag{23}$$

The reason is simple. The probability ϵ is the sum of the probabilities to go after one step in the points $\rho\pm\delta\rho$ or remain in ρ times the exit probabilities from these intermediate points.

Again by expanding to second order in $\delta\rho$ one obtains the backward Kolmogorov equation for the exit probability

$$v(\rho)\frac{d\epsilon(\rho)}{d\rho}+D(\rho)\frac{d^2\epsilon(\rho)}{d\rho^2}=0 \tag{24}$$

For voter, since $v(\rho)=0$ only the second term counts and the generic solution is $\epsilon(\rho)=A\rho+B$. Since the boundary conditions are $\epsilon(\rho=0)=0$ and $\epsilon(\rho=1)=1$, the solution is $\epsilon(\rho)=\rho$.

Consensus time In full analogy, one can write the equation for the average time to reach consensus $T(\rho)$, as given by the time to perform a step (δt) plus the time needed to reach consensus after the new step

$$T(\rho)=\delta t+R(\rho)T(\rho+\delta\rho)+L(\rho)T(\rho-\delta\rho)+[1-R(\rho)-L(\rho)]T(\rho) \tag{25}$$

Expanding to second order:

$$v(\rho)\frac{dT(\rho)}{d\rho}+D(\rho)\frac{d^2T(\rho)}{d\rho^2}=-1 \tag{26}$$

Specializing it to the voter case:

$$\frac{\rho(1-\rho)}{N}\frac{d^2T(\rho)}{d\rho^2}=-1 \tag{27}$$

The boundary conditions are $T(\rho=0)=T(\rho=1)=0$. Integrating one gets

$$T(\rho)=-N[\rho\ln\rho+(1-\rho)\ln(1-\rho)] \tag{28}$$

<u>*Comparison with Ising (1)*</u> For comparison, let us see what happens for Glauber $T = 0$ dynamics. On a complete graph, if we intend the dynamics in such a way that the local field of a node is due to all other nodes, things are particularly simple. The probabilities R and L are simply $R = \theta(\rho - 1/2)$ $L = \theta(1/2 - \rho)$, so that

$$v(\rho) = 1 - 2\theta(1/2 - \rho) \tag{29}$$

and the dynamics is dominated by drift. Diffusion is not important since $D(\rho)$ has a factor $1/N$ in front of it ad vanishes in the thermodynamics limit. Hence Eq.(24) becomes

$$v(\rho)\frac{d\epsilon(\rho)}{d\rho} = 0 \tag{30}$$

with boundary conditions $\epsilon(\rho = 0) = 0$ and $\epsilon(\rho = 1) = 1$, yielding

$$\epsilon(\rho) = \begin{cases} 0 & \rho < \frac{1}{2} \\ 1 & \rho > \frac{1}{2} \end{cases} \tag{31}$$

In the same way the equation for consensus time is

$$v(\rho)\frac{dT(\rho)}{d\rho} = -1 \tag{32}$$

implying

$$T(\rho) = \begin{cases} \rho & \rho < \frac{1}{2} \\ 1-\rho & \rho > \frac{1}{2} \end{cases} \tag{33}$$

<u>*Comparison with Ising (2)*</u> Another possibility is to intend the local field of Glauber dynamics as due to a finite number of randomly-chosen neighbors. In the case of only two randomly-chosen neighbors things are equal to the voter case. If instead one considers 4 neighbors:

$$R(\rho) = \rho^2(1-\rho)(3-2\rho) \tag{34}$$
$$L(\rho) = \rho(1-\rho)^2(1+2\rho) \tag{35}$$

so that

$$v(\rho) = R - L = \rho(1-\rho)(2\rho-1) \tag{36}$$

In the determination of the exit probability nothing changes: the diffusive term can be neglected and the solution is the step function as before.

For what concerns the consensus time

$$v(\rho)\frac{dT(\rho)}{d\rho} = -1 \Longrightarrow T = -\int_{1/N}^{\rho} \frac{d\rho'}{\rho'(1-\rho')(2\rho'-1)} \tag{37}$$

where one has to integrate from $1/N$ because velocity vanishes close to zero and one has to take $\rho < 1/2$ to avoid the same problem for $\rho = 1/2$. Performing the integral one finds

$$T \sim \log(N) + \log\left[\frac{\rho(1-\rho)}{(1-2\rho)^2}\right] \tag{38}$$

Asymptotically the consensus time grows as $\log(N)$ and does not depend on ρ. We find a behavior different from the voter case. In the voter case, the absence of a drift makes the convergence to consensus much slower than in the Glauber case.

D. Analytical solution in d-dimensions.

Considering a d-dimensional hypercubic lattice and denoting with $S = \{s_i\}$ the state of the system, the transition rate for a spin k to flip is

$$W_k(S) \equiv W(s_k \to -s_k) = \frac{d}{4}\left(1 - \frac{1}{2d}s_k\sum_j s_j\right), \tag{39}$$

The prefactor, setting the overall temporal scale, is chosen for convenience. The probability distribution function $P(S,t)$ obeys the master equation

$$\frac{d}{dt}P(S,t) = \sum_k \left[W_k(S^k)P(S^k,t) - W_k(S)P(S,t)\right], \tag{40}$$

where S^k is equal to S except for the flipped spin s_k.

Equation for the local magnetization Let us compute the equation for the local magnetization $\langle s_i\rangle \equiv \sum_S P(S,t)s_i$, by multiplying Eq. (40) by s_i and summing over S:

$$\frac{d}{dt}\sum_S P(S,t)s_i = \sum_S\sum_k \left[W_k(S^k)P(S^k,t) - W_k(S)P(S,t)\right] s_i, \tag{41}$$

Some useful relations are:

$$\sum_S f(s_j)P(S) = \sum_S f(s_j)P(S^k) \tag{42}$$

where $j \neq k$;

$$\sum_S s_k P(S) = \langle s_k\rangle \tag{43}$$

$$\sum_S s_k P(S^k) = -\langle s_k\rangle \tag{44}$$

The equation for the local magnetization is simply

$$\frac{d}{dt}\langle s_i\rangle = -2\langle s_i W_i\rangle \tag{45}$$

This formula can be understood by simply considering that when the spin i flips its magnetization changes by $-2s_i$.

A more precise demonstration: rates are linear in s_k:

$$\begin{aligned} W_k(S) &= W_k^0 + s_k W_k^1 && (46)\\ W_k(S^k) &= W_k^0 - s_k W_k^1 && (47) \end{aligned}$$

where W^0 and W^1 depend on the other spins but not on s_k. We insert these expressions into Eq. (41)

$$\begin{aligned} \frac{d}{dt}\langle s_i\rangle &= \sum_S\sum_k \left[(W_k^0 - s_k W_k^1)P(S^k,t) - (W_k^0 + s_k W_k^1)P(S,t)\right] s_i && (48)\\ &= \sum_S\sum_k \left[s_i W_k^0(P(S^k,t) - P(S,t)) - s_i s_k W_k^1(P(S^k,t) + P(S,t))\right] && (49) \end{aligned}$$

The first term is zero because of Eq. (42), except when $k = i$, when it is equal to $-2\langle s_i W_i^0\rangle$.

The second term is zero because of Eqs. (43) and (44), unless for $k = i$, when it is equal to 2 $\langle W_i^1\rangle$. Putting all together

$$\begin{aligned} \frac{d}{dt}\langle s_i\rangle &= -2\langle s_i W_i^0\rangle - 2\langle W_i^1\rangle && (50)\\ &= -2\langle s_i\left(W_i^0 + s_i W_i^1\right)\rangle && (51)\\ &= -2\langle s_i W_i\rangle && (52)\\ & && (53) \end{aligned}$$

hence,

$$\frac{d}{dt}\langle s_i\rangle = -2\frac{d}{4}\langle\left[s_i - \frac{1}{2d}s_i^2\sum_j s_j\right]\rangle \tag{54}$$

$$= +\frac{d}{2}\frac{1}{2d}\left[-2d\langle s_i\rangle + \langle s_i^2\sum_j s_j\rangle\right] \tag{55}$$

$$= \frac{1}{4}\Delta_i\langle s_i\rangle, \tag{56}$$

where the discrete laplacian is

$$\Delta_i\langle s_i\rangle = \left[-2d\langle s_i\rangle + \sum_j\langle s_j\rangle\right] \tag{57}$$

If we sum over i and divide by the total number of sites, we obtain that the average total magnetization $\langle s\rangle = \frac{1}{N}\sum_i\langle s_i\rangle$ is conserved

$$\frac{d}{dt}\langle s\rangle = 0 \tag{58}$$

Notice that the magnetization is conserved on average, not for every single dynamical event.

Exit probability An immediate consequence of this conservation is that the exit probability is derived straightforwardly. Consider an initial magnetization $s(0)$, corresponding to an initial fraction of up spins $\rho(0) = [1 + s(0)]/2$. Conservation of the magnetization implies that also in the final state the fraction of up spins must be equal to $\rho(0)$. But the final state can be of only two types, either all spins up or all spins down. Hence, calling $E[\rho(0)]$ the exit probability, i.e. probability to end in the state with all spins up,

$$\rho(0) = +1 \times E[\rho(0)] + 0 \times (1 - E[\rho(0)]) \tag{59}$$

from which

$$E[\rho(0)] = \rho(0) \tag{60}$$

The probability is linear in the initial magnetization (it is equal to the initial fraction of up spins).

Two point correlations In a similar manner correlation functions of higher order are derived. In particular, the second order correlation function $\langle s_{k,l}\rangle = \sum_S P(S)s_k s_l$ obeys

$$\frac{d}{dt}\langle s_{k,l}\rangle = \frac{1}{4}(\Delta_k + \Delta_l)\langle s_{k,l}\rangle \tag{61}$$

It does not depend on higher order correlation functions. The usual hierarchy is interrupted. The equation is closed. This is the key point leading the the exact solvability of the model.

Using these expressions one finds the asymptotic long time limit of the density of active interfaces $n_{AB} = [1 - C(r = 1, t)]/2$

$$n_{AB}(t) = \begin{cases} t^{-1+d/2} & d < 2 \\ 1/\ln(t) & d = 2 \\ a - bt^{-d/2} & d > 2 \end{cases} \tag{62}$$

Notice that for $d > 2$ this quantity does not go to zero: the system remains disordered.

In the marginal case of two-dimensions $n_{AB}(t)$ can be computed more precisely

$$n_{AB}(t) = \frac{\pi}{2\ln(t) + \ln(256)} + O\left(\frac{\ln(t)}{t}\right) \tag{63}$$

One can even go further and derive the full scaling expression of the correlation function for the voter model in $2d$:

$$C(r,t) = \frac{1}{\ln(16t)}E_1[r^2/(2t))] \tag{64}$$

where $E_1(x)$ is the exponential integral function $E_1(x) = \int_z^\infty \frac{e^{-u}}{u}du$. This expression is different from the standard scaling expression $C(r,t) = F(r/L(t))$, indicating that standard scaling is violated: there are *two* distinct lengthscales in the expression, not only $L(t) = t^{1/2}$. A more correct physical interpretation is that there are *infinitely* many lengths in the system: there are domains of all sizes up to $L(t)$.

<u>*Generality of voter behavior*</u> It is possible to study models which interpolate between Glauber and Ising in $d = 2$. One is the model by De Oliveira et al. [3], where there is one free parameter, corresponding to the probability that a spin surrounded by three opposite spins becomes equal to them The voter model corresponds to such probability equal to 3/4. For values larger than 3/4 one is in a ferromagnetic ordered phase, where the ordering dynamics is the one predicted by Glauber zero temperature dynamics. For values smaller than 3/4 one is in the disordered phase of a ferromagnetic Glauber-like dynamics. Voter dynamics sits exactly at the transition point, hence it is critical.

It could seem that the voter model is rather peculiar, being a singular point in the phase-diagram.

There are many other works [4, 5] showing that minimal changes in the dynamics completely destroy voter-like behavior. They raise the question of whether the voter model is a peculiar exception or the representative of a more generic class of models.

An answer to this question is provided by Dornic *et al.* [6]. The authors of this work show explicitly the existence of models, different from the pure linear voter, that nevertheless exhibit its typical dynamical features. This leads to the conjecture that there is a proper generalized voter (GV) universality class encompassing systems at "an order-disorder transition driven by the interfacial noise between two absorbing states possessing equivalent dynamical roles, this symmetry being enforced either by Z_2 symmetry of the local rules, or by the global conservation of the magnetization" [6].

Further progress in the understanding of this issue has been made by Al Hammal *et al.* [7], who worked out a generic Langevin equation for critical phenomena with two symmetric absorbing states and identified conditions for having a transition from order to disorder belonging to the GV class.

$$\partial_t \phi = (a\phi - b\phi^3)(1 - \phi^2) + D\nabla^2\phi + \sigma\sqrt{1 - \phi^2}\eta \tag{65}$$

Note that in voter-like models there are two different competing phenomena: One is the breaking of the Z_2 symmetry and the other one is the possibility for the system to get trapped into an absorbing state. If both occur in unison then the transition point is in the GV class. Instead, if they occur separately, the Z_2 symmetry is broken first (i.e. an Ising like transition occurs, and the system changes from paramagnetic to ferromagnetic) and afterwards the system falls into the corresponding absorbing state (i.e a directed percolation like transition) [7]. In this sense, the GV class can be rationalized as the superposition of Ising and directed percolation phase transitions.

The picture devised in Ref. [7] on the basis of generic symmetry arguments has been recently substantiated by Vázquez and López [8]. Starting from the microscopic spin dynamics of a *non-linear voter model*, they have derived an explicit Langevin equation for the magnetization, that coincides with the one conjectured in Ref. [7]. In this way, it is possible to precisely determine, depending on the analytical form of the microscopic flipping probability $f(x)$, which of the two scenarios above occurs.

E. Modifications of voter model

<u>*AB model*</u> Another modification is to allow for 'mixed states' to exist. A similar model with three states is the AB-model [4], interpreted as a model for bilingualism. Here the state of an agent evolves according to the following rules. At each time step one randomly selects an agent i and updates its state according to the following transition probabilities, depending on the densities ρ_A, ρ_B and ρ_{AB} of each state in the neighborhood of i.

$$p_{A\to AB} = 1/2\rho_B, \qquad p_{B\to AB} = 1/2\rho_A, \tag{66}$$

$$p_{AB\to B} = 1/2(1-\rho_A), \quad p_{AB\to A} = 1/2(1-\rho_B), \tag{67}$$

The idea here is that, in order to go from A to B one has to pass through the intermediate state AB. At odds with other models (constrained voter model) here extremes do interact, since the rate to go from state A to AB is proportional to the density of neighbors in state B. This implies that consensus on the AB state or a frozen mixture of A and B is not possible, the only two possible absorbing states being those of consensus of A or B type.

It turns out that the possibility of having a bilingual state reintroduces surface tension, so that the behavior becomes similar to the Glauber dynamics.

F. Axelrod model

A prominent role in the investigation of cultural dynamics has been played by a model introduced by Axelrod in [9], that has attracted a lot of interest from both social scientists and physicists.

The origin of its success among social scientists is in the inclusion of two mechanisms that are believed to be fundamental in the understanding of the dynamics of cultural assimilation (and diversity): social influence and

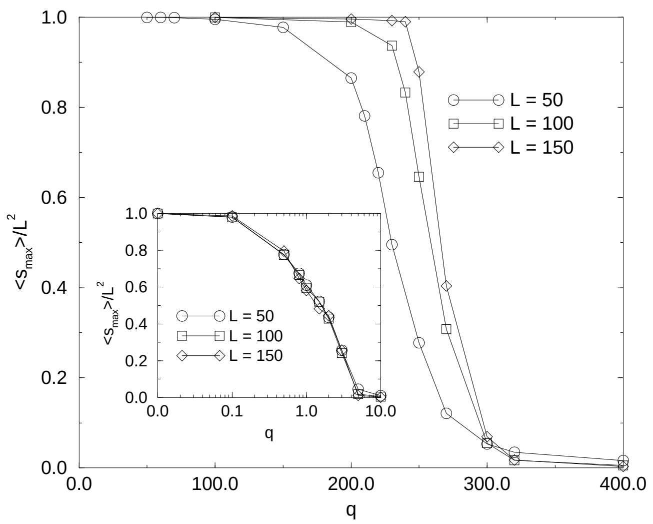

FIG. 5: Axelrod model. Behavior of the order parameter $\langle S_{max}\rangle/L^2$ vs. q for three different system sizes and $F = 10$. In the inset the same quantity is reported for $F = 2$. From [10].

homophily. The first is the tendency of individuals to become more similar when they interact. The second is the tendency of likes to attract each other, so that they interact more frequently. These two ingredients were generally expected by social scientists to generate a self-reinforcing dynamics leading to a global convergence to a single culture. It turns out instead that the model predicts in some cases the persistence of diversity.

From the point of view of statistical physicists, the Axelrod model is a simple and natural "vectorial" generalization of the voter model, that gives rise to a very rich and nontrivial phenomenology, with some genuinely novel behavior. The model is defined as follows. Individuals are located on the nodes of a network (or on the sites of a regular lattice) and are endowed with F integer variables $(\sigma_1, \ldots, \sigma_F)$ that can assume q values, $\sigma_f = 0, 1, ..., q-1$. The variables are called cultural *features* and q is the number of the possible *traits* allowed per feature. They are supposed to model the different "beliefs, attitudes and behavior" of individuals. In an elementary dynamic step, an individual i and one of his neighbors j are selected and the overlap between them

$$\omega_{i,j} = \frac{1}{F}\sum_{f=1}^{F} \delta_{\sigma_f(i),\sigma_f(j)}, \tag{68}$$

is computed, where $\delta_{i,j}$ is Kronecker's delta. With probability $\omega_{i,j}$ the interaction takes place: one of the features for which traits are different $(\sigma_f(i) \neq \sigma_f(j))$ is selected and the trait of the neighbor is set equal to $\sigma_f(i)$. Otherwise nothing happens. It is immediately clear that the dynamics tends to make interacting individuals more similar, but the interaction is more likely for neighbors already sharing many traits (homophily) and it becomes impossible when no trait is the same. There are two stable configurations for a pair of neighbors: when they are exactly equal, so that they belong to the same cultural region or when they are completely different, i.e., they sit at the border between cultural regions.

Starting from a disordered initial condition (for example with uniform random distribution of the traits) the evolution on any finite system leads unavoidably to one of the many absorbing states, which belong to two classes: the q^F ordered states, in which all individuals have the same set of variables, or the other, more numerous, frozen states with coexistence of different cultural regions.

An applet with the Axelrod model can be found at `http://ifisc.uib-csic.es/research_topics/socio/culture.html`

It turns out that which of the two classes is reached depends on the number of possible traits q in the initial condition [10]. For small q individuals share many traits with their neighbors, interactions are possible and quickly full consensus is achieved. For large q instead, very few individuals share traits. Few interactions occur, leading to the formation of small cultural domains that are not able to grow: a disordered frozen state. On regular lattices, the two regimes are separated by a phase transition at a critical value q_c, depending on F (Fig. 5). Several order parameters can be defined to characterize the transition. One of them is the average fraction $\langle S_{max}\rangle/N$ of the system occupied by

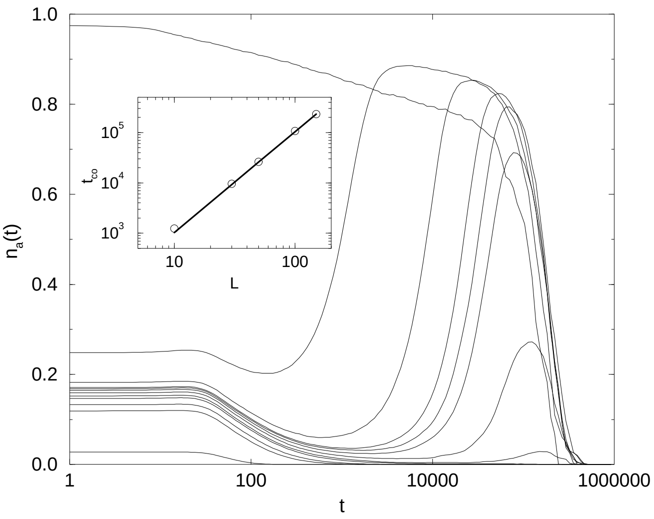

FIG. 6: Plot of the density of active links $n_a(t)$ for $F = 10$, $L = 150$ and (top to bottom) $q = 1, 100, 200, 230, 240, 250, 270, 300, 320, 400, 500, 10000$. The inset reports the dependence of the freezing time t_{co} on L for $F = 10$ and $q = 100 < q_c$. The bold line has slope 2. From [10].

the largest cultural region. N is the number of individuals in the system. In the ordered phase this fraction is finite (in the limit $N \to \infty$), while in the disordered phase cultural domains are of finite size, so that $\langle S_{max} \rangle / N \sim 1/N$. Another (dis)order parameter often used [11] is $g = \langle N_g \rangle / N$, where N_g is the number of different domains in the final state. In the ordered phase $g \to 0$, while it is finite in the disordered phase.

In two dimensions the nature of the transition depends on the value of F. For $F = 2$ there is a continuous change in the order parameter at q_c, while for $F > 2$ the transition is discontinuous (Fig. 5). Correspondingly the distribution of the size s of cultural domains at the transition is a power law with $P(s) \sim s^{-\tau}$ exponent smaller than 2 ($\tau \approx 1.6$) for $F = 2$ while the exponent is larger than 2 ($\tau \approx 2.6$) for any $F > 2$. In one-dimensional systems instead [12], the transition is continuous for all values of F.

It is worth remarking that, upon interaction, the overlap between two neighbors always increases by $1/F$, but the change of a trait in an individual can make it more dissimilar with respect to his other neighbors. Hence, when the number of neighbors is larger than 2, each interaction can, somewhat paradoxically, result in an increase of the general level of disorder in the system. This competition is at the origin of the nontrivial temporal behavior of the model in $d = 2$, illustrated in Fig. 6: below the transition but close to it ($q \lesssim q_c$) the density of active links (connecting sites with overlap different from 0 and 1) has a highly non monotonic behavior. Most investigations of the Axelrod model are based on numerical simulations of the model dynamics. Analytical approaches are just a few.

G. Majority rule

In the Majority Rule model (MR), each individual is endowed with a binary variable. At each time step a group of r individuals is considered and all are made equal to the state held by the majority inside the group [13].

The model was introduced by Galam with r a random number with prescribed distribution and emphasis on what happens when r is even and in the case of a tie.

More interesing is the case of fixed odd r. In MF the solution can be derived both for a finite population of N agents and in the continuum limit of $N \to \infty$ as done previously for voter and Glauber.

In the case of the majority rule, the rates R and L that appear in the expression of the Fokker-Planck equation are, for $r = 3$

$$R(\rho) = \rho^2(1-\rho) \quad (69)$$
$$L(\rho) = \rho(1-\rho)^2 \quad (70)$$

The expression for the velocity v is then

$$v(\rho) = \rho(1-\rho)(2\rho-1) \tag{71}$$

which coincides with the expression (36) for Glauber. As a consequence, the exit probability and the consensus time are given by the same expressions: the exit probability is a step function and the consensus time is (asymptotically) proportional to $\ln N$ and not depending on ρ.

On a d-dimensional lattice, the discussion group is localized around a randomly chosen lattice site. In one dimension, the model is not analytically solvable. Since the average magnetization is not conserved by the MR dynamics, the exit probability, i.e., the probability that the final magnetization is +1, has a non-trivial dependence on the initial magnetization in the thermodynamic limit and a minority can actually win the contest. Consensus time grows as N^2. In higher dimensions, the dynamics is characterized by diffusive coarsening. When the initial magnetization is zero, the system may be trapped in metastable states (stripes in $2d$, slabs in $3d$), which evolve only very slowly. This leads to the existence of two distinct temporal scales: the most probable consensus time is short but, when metastable states appear, the time needed is exceedingly longer. As a consequence, the average consensus time grows as a power of N, with a dimension-dependent exponent. When the initial magnetization is non-zero, metastable states quickly disappear. A crude coarse-graining argument reproduces qualitatively the occurrences of metastable configurations for any d.

H. Bounded confidence

In the models we have so far investigated opinion is a discrete variable. However, there are cases in which the position of an individual can vary smoothly from one extreme to the other of the range of possible choices.

The initial state is usually a population of N agents with randomly assigned opinions, represented by real numbers within some interval. In principle, each agent can interact with every other agent, no matter what their opinions are. In practice, there is a real discussion only if the opinions of the people involved are sufficiently close to each other. This realistic aspect of human communications is called *bounded confidence* (BC); in the literature it is expressed by introducing a real number ϵ, the *uncertainty* or *tolerance*, such that an agent, with opinion x, only interacts with those of its peers whose opinion lies in the interval $]x-\epsilon, x+\epsilon[$.

We discuss the most popular BC model, i.e., the Deffuant model.

Let us consider a population of N agents, represented by the nodes of a graph, where agents may discuss with each other if the corresponding nodes are connected. Each agent i is initially given an opinion x_i, randomly chosen in the interval $[0,1]$. The dynamics is based on random binary encounters, i.e., at each time step, a randomly selected agent discusses with one of its neighbors on the social graph, also chosen at random. Let i and j be the pair of interacting agents at time t, with opinions $x_i(t)$ and $x_j(t)$, respectively. Deffuant dynamics is summarized as follows: if the difference of the opinions $x_i(t)$ and $x_j(t)$ exceeds the threshold ϵ, nothing happens; if, instead, $|x_i(t) - x_j(t)| < \epsilon$, then:

$$x_i(t+1) = x_i(t) + \mu[x_j(t) - x_i(t)], \tag{72}$$

$$x_j(t+1) = x_j(t) + \mu[x_i(t) - x_j(t)]. \tag{73}$$

The parameter μ is the so-called convergence parameter, and its value lies in the interval $[0, 1/2]$. Deffuant model is based on a compromise strategy: after a constructive debate, the positions of the interacting agents get closer to each other, by the relative amount μ. If $\mu = 1/2$, the two agents will converge to the average of their opinions before the discussion. For any value of ϵ and μ, the average opinion of the agents' pair is the same before and after the interaction, so the global average opinion (1/2) of the population is an invariant of Deffuant dynamics.

The evolution is due to the instability of the initial uniform configuration near the boundary of the opinion space. Such instability propagates towards the middle of the opinion space, giving rise to patches with an increasing density of agents, that will become the final opinion clusters. Once each cluster is sufficiently far from the others, so that the difference of opinions for agents in distinct clusters exceeds the threshold, only agents inside the same cluster may interact, and the dynamics leads to the convergence of the opinions of all agents in the cluster to the same value. Therefore, the final opinion configuration is a succession of Dirac's delta functions. In general, the number and size of the clusters depend on the threshold ϵ, whereas the parameter μ affects the convergence time of the dynamics. However, when μ is small, the final cluster configuration also depends on μ.

On complete graphs, regular lattices, random graphs and scale-free networks, for $\epsilon > \epsilon_c = 1/2$, all agents share the same opinion 1/2, so there is complete consensus. This may be a general property of Deffuant model, independently

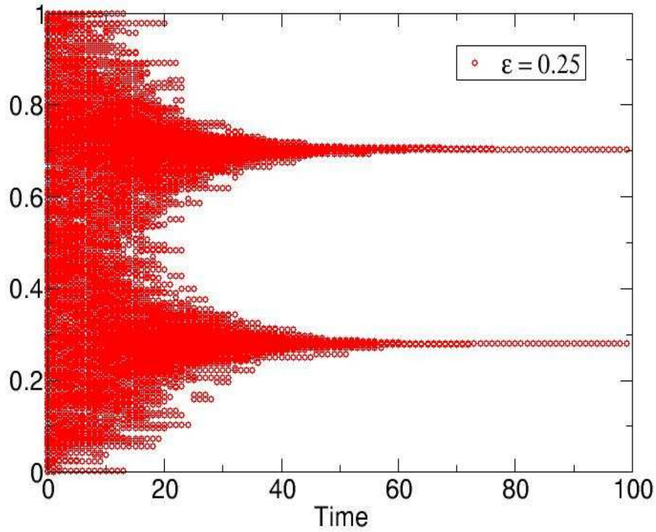

FIG. 7: Deffuant model. Opinion profile of a population of 500 agents during its time evolution, for $\epsilon = 0.25$. The population is fully mixed, i.e., everyone may interact with everybody else. The dynamics leads to a polarization of the population in two factions.

of the underlying social graph. If ϵ is small, more clusters emerge (Fig. 7). Monte Carlo simulations reveal that the number n_c of clusters in the final configuration can be approximated by the expression $1/(2\epsilon)$. This can be understood if we consider that, at stationarity, agents belonging to different opinion clusters cannot interact with each other, which means that the opinion of each cluster must differ by at least ϵ from the opinions of its neighboring clusters. In this way, within an interval of length 2ϵ centered at a cluster, there cannot be other clusters, and the ratio $1/(2\epsilon)$ is a fair estimate for n_c.

Most results on Deffuant dynamics are derived through numerical simulations, as the model is not analytically solvable. However, in the special case of a fully mixed population, where everybody interacts with everybody else, it is possible to write the general rate equation governing the opinion dynamics [14]. For this purpose, one neglects individual agents and focuses on the evolution of the opinion population $P(x,t)$, where $P(x,t)dx$ is the probability that an agent has opinion in the interval $[x, x+dx]$. The interaction threshold is $\epsilon = 1$, but the opinion range is the interval $[-\Delta, \Delta]$; this choice is equivalent to the usual setting of the Deffuant model, if $\epsilon = 1/2\Delta$. For simplicity, $\mu = 1/2$. The rate equation then reads:

$$\frac{\partial}{\partial t}P(x,t) \;=\; \int_{|x_1-x_2|<1}\!\!\int dx_1 dx_2 P(x_1,t)P(x_2,t) \times \left[\delta\Big(x - \frac{x_1+x_2}{2}\Big) - \delta(x-x_1)\right].$$

Eq. (74) conserves the norm $M_0 = \int_{-\Delta}^{+\Delta} P(x,t)dx = 2\Delta$ and the average opinion. The question is to find the asymptotic state $P_\infty(x) = P(x, t\to\infty)$, starting from the flat initial distribution $P(x,t=0) = 1$, for $x \in [-\Delta, \Delta]$. If $\Delta < 1/2$, all agents interact and Eq. (74) is integrable. In this case, it is possible to show that all agents approach the central opinion 0 and $P_\infty(x) = M_0\delta(x)$ and determine the temporal evolution of the second moment $M_2(t)$, which decreases exponentially over time.

If $\Delta > 1/2$, the equation is no longer analytically solvable. The asymptotic distribution is a linear combination of delta functions, separate by a distance larger than 1, with masses m_i

$$P_\infty(x) = \sum_{i=1}^{p} m_i \delta(x - x_i). \tag{74}$$

The cluster masses m_i must obey the conditions $\sum_i m_i = M_0$ and $\sum_i m_i x_i = 0$; the latter comes from the conservation of the average opinion.

Numerical solutions of Eq. (74) reveal that, as Δ is increased, there is a complicated succession of bifurcation transition, with only three types of clusters: major (mass > 1), minor (mass $< 10^{-2}$) and a central cluster located

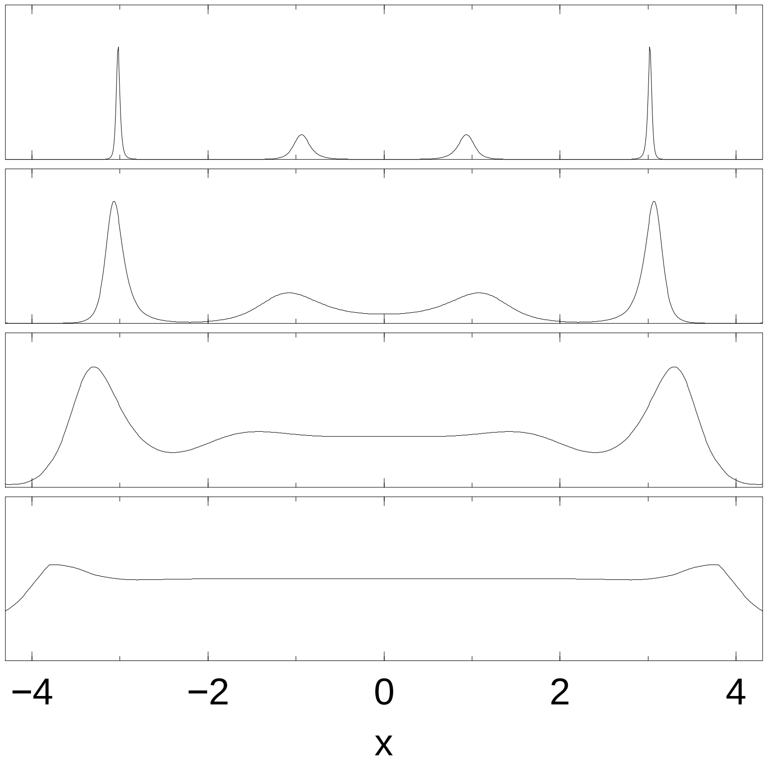

FIG. 8: Evolution of the opinion distribution for $\Delta = 4.3$ where four major clusters ultimately arise. Shown is $P(x,t)$ versus x for times $t = 0.5$ (bottom), 3, 6, and 9 (top).

at $x = 0$. These clusters are generated by a periodic sequence of bifurcations, consisting in the nucleation and annihilation of clusters.

III. COLLECTIVE OPINION SHIFTS AND THE RANDOM FIELD ISING MODEL

We present here a nice example of the application of a statistical physics model to social phenomena. For more details, see Ref. [15]. Consider N individuals. Each agent i is confronted with a binary choice $s_i = \pm 1$. It can be yes/no in a referendum, to buy/not to buy a certain good, to clap/not to clap in an audience.....

The decision of agent i depends on three distinct factors:

- his personal attitudes (or utility), measured by a real time independent variable $-\infty < \phi_i < \infty$; large ϕ_i means propensity to be in state $s_i = +1$. Call $p(\phi)$ the p.d.f. of the fields ϕ_i.
- public information, affecting all agents equally (price, technological advances) $-\infty < F(t) < \infty$;
- social pressure or imitation effects. Each agent i is influenced by each other agent j in a certain neighborhood V_i via a coupling $J_{i,j}S_i$ that is summed to ϕ_i and F. The couplings are taken positive. If they are negative complications may arise.

The rule for deciding the state of agent i at time t is simply to see whether the sum of the three factors is positive or negative;

$$s_i(t) = \text{sign}\left[\phi_i + F_i(t) + \sum_{j \in V_i} J_{i,j} S_j(t-1)\right] \tag{75}$$

Any other choice of the threshold amounts to a rescaling of the fields ϕ_i.

But in physics this is nothing else than the zero temperature dynamics of the Random Field Ising Model.

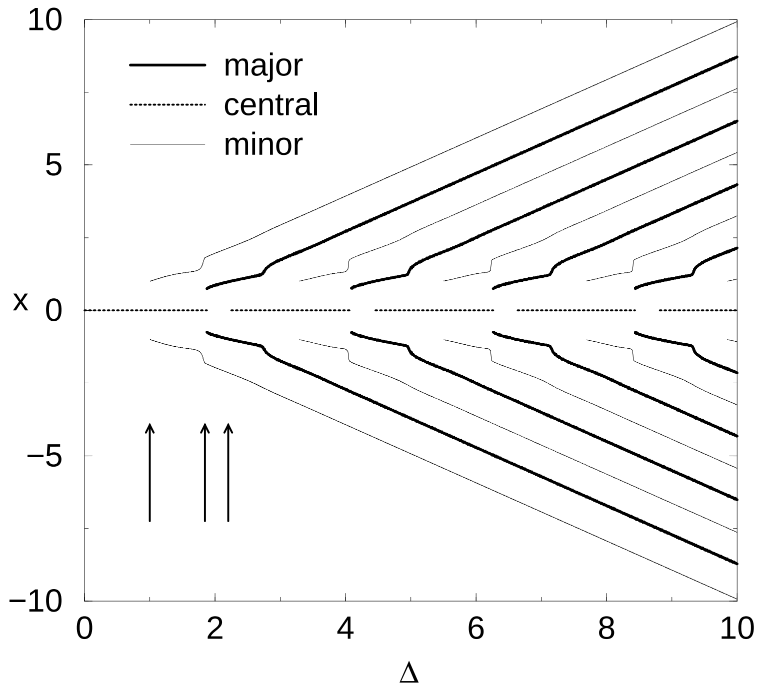

FIG. 9: Location of final state clusters versus the initial opinion range Δ. The three types of clusters, defined in the text, are noted. The vertical arrows indicate the location of the first 3 bifurcations

A. No social pressure (non interacting agents)

Consider $J_{i,j} \equiv 0$. Each individual is non-interacting.

We sum Eq. (75) over i and divide by the number of agents N, obtaining an equation for the average opinion (magnetization) $M = 1/N \sum_i s_i$.

$$M_0 = \frac{1}{N} \sum_i \mathrm{sign}[\phi_i + F] \tag{76}$$

$$= -\int_{-\infty}^{-F} d\phi p(\phi) + \int_{-F}^{\infty} d\phi p(\phi) \tag{77}$$

$$= -R(-F) + [1 - R(-F)] = 1 - 2R(-F) \tag{78}$$

As F grows from $-\infty$ to ∞, the fraction of agents in state $+1$ increases smoothly reflecting the distribution of a priori opinions in the population.

B. Mean-field analysis

Let us consider the presence of imitation in the mean-field case: $J_{i,j} = J/N$.

This does not mean that each agent consults all the others, rather that the average opinion becomes public opinion, thus influencing the opinion of each individual.

The effect of the interaction is to shift $F \to F + MJ$, leading to the self-consistent equation:

$$M = 1 - 2R(-F - JM) \tag{79}$$

If imitation (J) is weak one can expand the right hand side

$$1 - 2R(-F - JM) = 1 - 2R(-F) + 2\frac{dR}{d\phi}|_{\phi=-F} JM = 1 - 2R(-F) + 2p(-F)JM \tag{80}$$

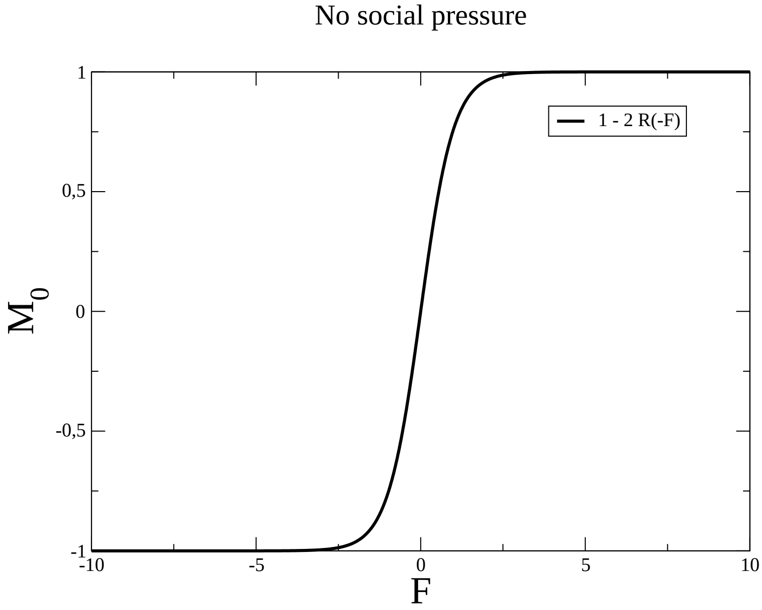

FIG. 10: System with no social pressure: $M_0 = 1 - 2R(-F)$ vs F.

where $p(-F)$ is the probability density of the local fields. Hence

$$M[1 - 2p(-F)J] = 1 - 2R(-F) = M_0 \Rightarrow \qquad M = \frac{M_0}{1 - 2p(-F)J} \tag{81}$$

This equation shows that around the point where $p(-F)$ is maximum imitation leads to an amplification of the opinion changes.

What happens for larger values of J? One has to solve self-consistently Equation (79).

Assume for simplicity a normal distribution for the fields ϕ:

$$p(\phi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\phi^2/(2\sigma^2)\right] \tag{82}$$

Assume for the moment that $F = 0$.

The graphical solution of Eq. (79) is (see Fig. 11)

Whether there are three solutions or just one depends on the slope of $1 - 2R(-JM)$ at $M = 0$ is larger or smaller than 1.

$$1 - 2R(-JM) = 1 - 2R(0) + 2p(0)JM.... = 2p(0)JM.... \tag{83}$$

Hence for $2p(0)J > 1$ there are three solutions, otherwise there is only one. This defines a critical threshold $J_c = 1/[2p(0)] = \sqrt{\pi/2}\sigma$ separating the two types of behavior.

If F is different from zero the curve in the figure is shifted so that, beyond some limits there is only one curve.

For generic distribution $p(\phi)$ the computations are more complicated but J_c is always proportional to the variance of the distribution of the fields ϕ. This makes sense: when all fields are the same $J_c = 0$ there is a single large avalanche. When the variance is large $J < J_c$ and each agent switches at a value of F close to its own ϕ.

As a function of F the behavior is then the one represented in Fig. 12. Above J_c there are three solutions one of which being unstable.

As F is increased from $-\infty$, for $J > J_c$ the average opinion follows the lower branch until it jumps discontinuously to the upper branch at a certain threshold field $F_c(J)$ (and symmetrically on the way back at a field $-F_c(J)$).

This behavior, that in magnetic terms indicates the presence of hysteresis, in social terms means that a population as a whole may undergo sudden, apparently irrational opinion swings, triggered by an infinitesimal external variation.

In economic context, the demand for a product can vary discontinuously as the price is decreased.

These discontinuities are absent in traditional models as the Bass model. [In Bass model there is no threshold: the probability of purchasing a new product is a function of the amount of others that already have bought it. It gives a sigmoidal function $M(t)$].

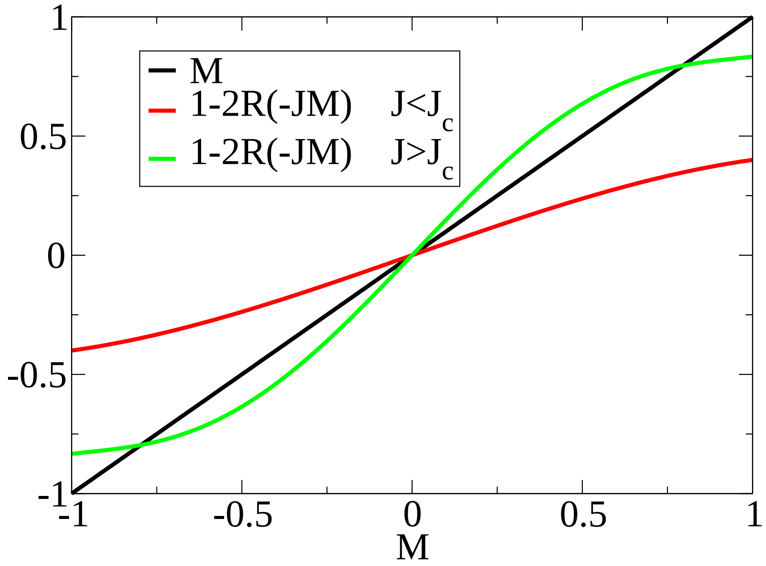

FIG. 11: Graphical solution of Eq. (79)

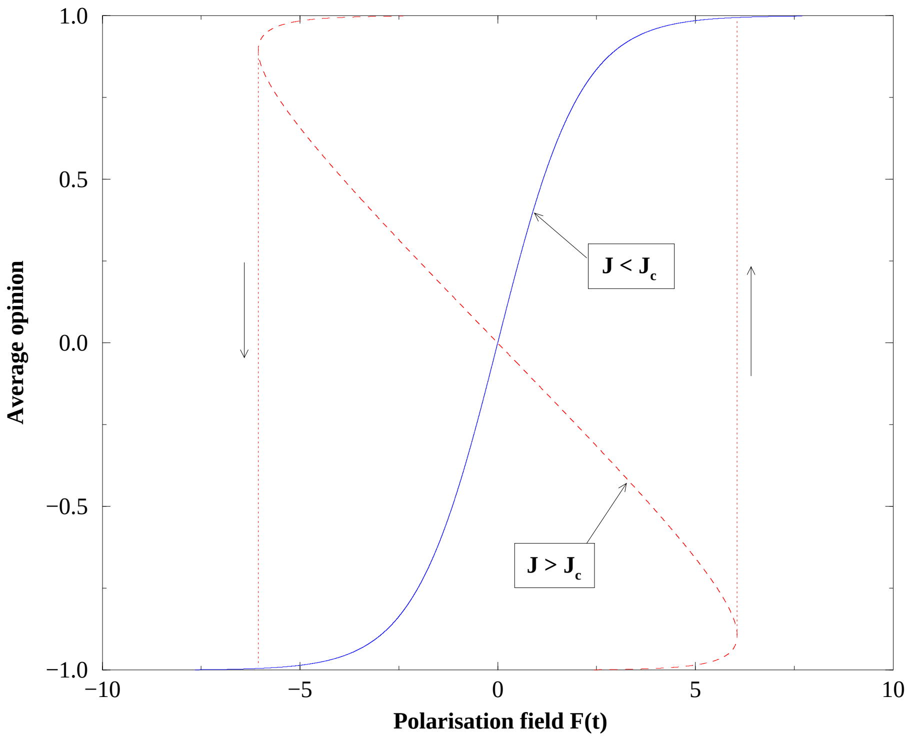

FIG. 12: Average opinion as a function of the external solicitation field $F(t)$. For an imitation parameter $J < J_c$, the curve is smooth; whereas for $J > J_c$ an hysteresis effect appears: the average opinion remains small for an anomalously large value of $F(t)$ before suddenly jumping to the upper branch.

Vicinity of critical point In the vicinity of the critical point (but for $J < J_c$) there are interesting critical phenomena, largely independent of the detailed form of $p(F)$. In particular it can be computed explicitly the behavior of the 'opinion slope' dM/dF as a function of the external field F and of the distance from criticality $\epsilon = J_c - J$

$$\frac{dM}{dF} = \frac{1}{\epsilon} \; G\left(\frac{F - F_c(J)}{\epsilon^{3/2}}\right) \tag{84}$$

The scaling function G is universal: $G(0)$ is a finite constant and $G(x \to \infty) \sim x^{-2/3}$. This means that, as a function of F the slope peaks at a maximum of order $1/\epsilon$ and remains large in a small window of order $w \sim \epsilon^{3/2}$. As a consequence the height h of the peak scales with the width w as

$$h \sim w^{-\kappa} \tag{85}$$

with exponent $\kappa = 2/3$, as opposed to models without the threshold effect (like Bass), where $\kappa = 1$.

The exponent $\kappa = 2/3$ is due to the presence of social pressure.

C. Robustness of MF results

- The critical regime in the RFIM is known to be quite large. Therefore the assumption to be in the critical regime is plausible.
- The distribution of the fields ϕ is irrelevant.
- The topology of the graph is irrelevant. The function G and the exponent κ change only if agents are on a regular lattice of dimension less than $d_c = 6$ (upper critical dimension of RFIM).

D. Comparison with real-world data

The model predicts

- Qualitatively different behaviors depending on the value of J.
- Quantitative predictions for the scaling around the critical point.

We now look for this kind of regularities in different real-world systems. The idea is that different countries or different crowds are characterized by different values of J or σ (and hence J_c) so that they will be at different distances from the critical point and hence at different values of ϵ.

1. Birth rates in Europe, 1950-2000

After World War II birth rates in Europe have switched from high level to low levels, with rather sudden drops (Fig. 13). The evolution of $F(t)$ is given by artificial birth control, higher education, loss of influence of religions.

Data are quite noisy but discrete derivatives can be taken and dM/dt can be fitted with a Gaussian to determine height, width and location of the peak. There is a large variability of the values of w and h for the various countries, but the best fit of the scaling (Fig. 14) is for $\kappa = 0.71$, closer to 2/3 than to 1.

2. Number of cell phones in Europe, 1994-2003

In several european countries the number of cell phones increased by a factor 20 or larger between 1994 and 2003. Here $F(t)$ is given by technological advances and reduced prices. Monthly evolution of the total number of cell phones in use (Fig. 15)

Germany has the the most collective behavior. Italy the less collective. The best fit gives a value of $\kappa = 0.62$ (see Fig. 14). Again the exponent is close to 2/3.

3. Clapping dynamics

In clapping there are interesting collective effects leading to the synchronization of claps. Here we focus on the way clapping dies out. Here the field ϕ_i is the a priori amount of time a given individual would carry on clapping if isolated. It reflects the different degrees of enthousiasm about the performance. Social influence comes from the fact that if we hear other clapping we tend to clap. If others stop we do not like being the last ones to clap. The average opinion here is the clapping intensity that goes from high to zero (Fig. 16).

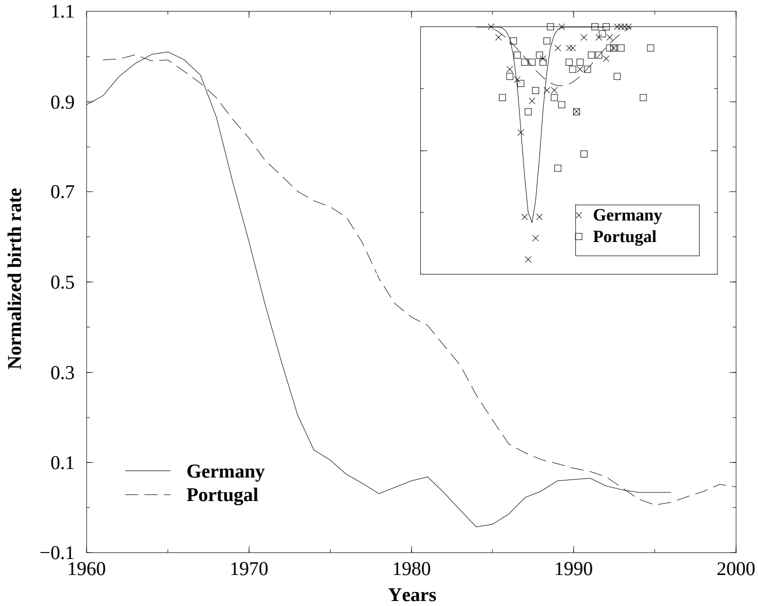

FIG. 13: Normalized fecundity index as a function of time for Germany and Portugal (3 year average). Other countries are intermediate in terms of the sharpness of the crossover. Inset: Yearly change of the fecundity index and Gaussian fits, allowing one to extract both the height h and width w of the peaks.

Again we find that the scaling of the width with the height of the peak is compatible with $\kappa = 2/3$ (Fig. 17).

But interestingly, here there are two instances where the clapping stops suddenly, on a temporal scale smaller than the acoustic decay time of the room (Fig. 18).

This can be interpreted, in terms of the RFIM model as the macroscopic avalanche that occurs for $J > J_c$.

Thus the decay of clappings shows two different types of behavior, both of which are well described by the RFIM, for different values of J and J_c.

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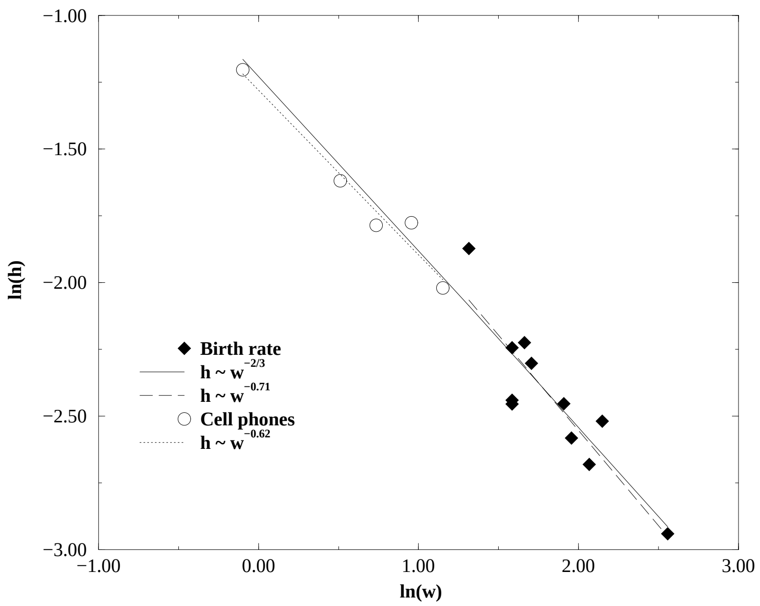

FIG. 14: Height of the peak h, vs. width of the peak w, in a log-log scale, both for birth rates and for cell phones. The mean field prediction $h \sim w^{2/3}$ is shown for comparison. A typical relative error of 20% on the fitted values of h or w translates into vertical and horizontal error bars of 0.2, comparable with the erratic spread of the points. The heights corresponding to cell phone data has been divided by a factor 1.7 to match the birth rate data (the absolute height of the peak is in fact not universal). The width w is however not rescaled, and shows that the explosion of cell phones is, as expected, faster that the collapse of the birth rate.

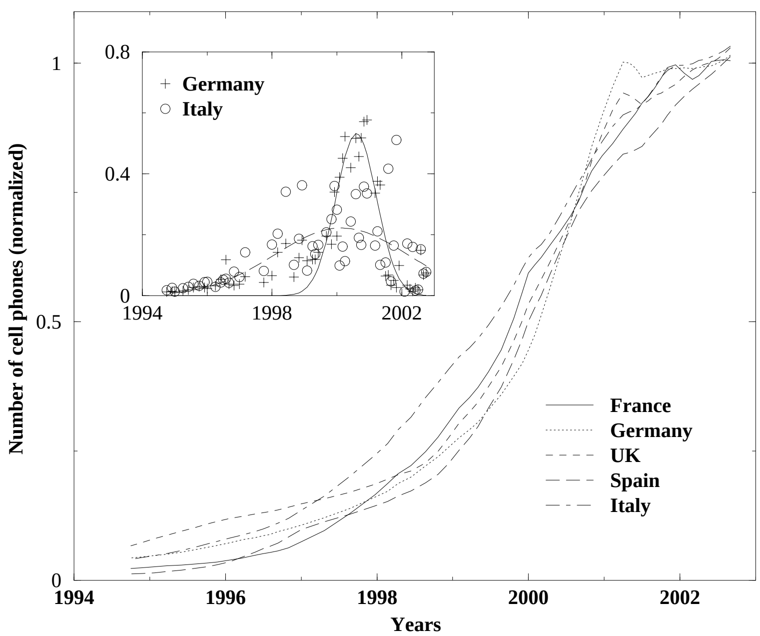

FIG. 15: Evolution of the total number of cell phones in use (all providers included) in various European countries (3 month average). Inset: Monthly change for Germany and Italy, allowing one to extract both the height h and width w of the peaks.

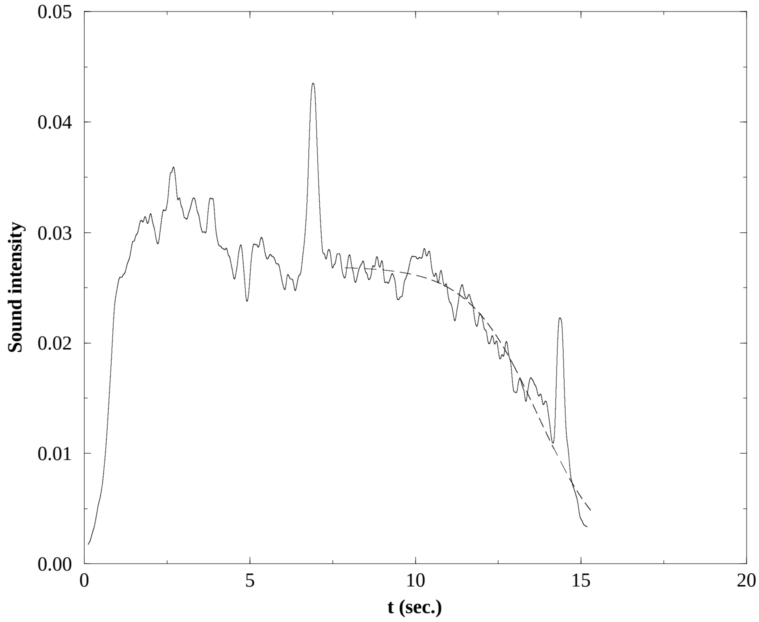

FIG. 16: Typical time series of sound intensity as a function of time, during applause. One sees the initial rise, a relatively constant plateau phase, and the final phase where clapping is tapering off. Notice a few spurious spikes, corresponding to occasional bravos or other shouts. The data was filtered with a Gaussian window of width 0.225 sec.

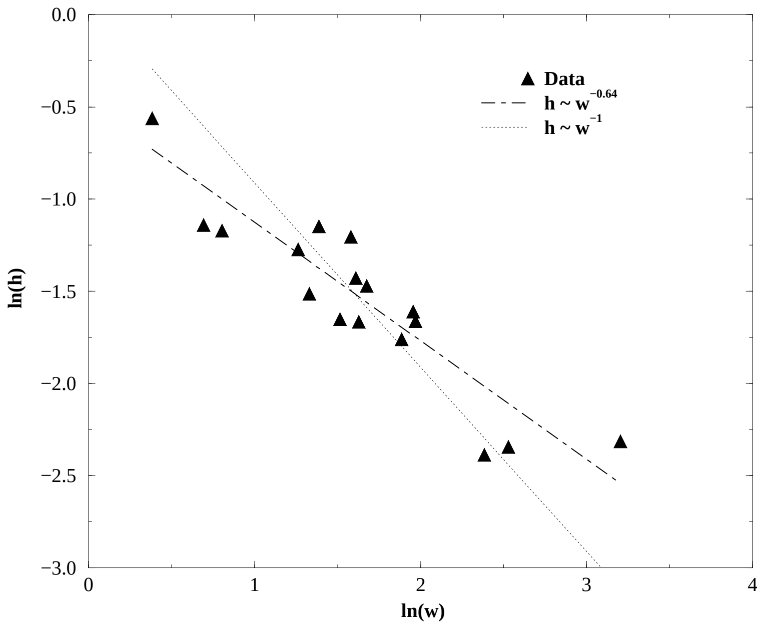

FIG. 17: Height of the peak h, vs. width of the peak w, in a log-log scale, for 17 applause endings. A best fit (in log scale) leads to a slope of 0.640.07, again very close to the mean-field RFIM prediction of $2/3$. Note that the above value of the error bar comes only from the regression. We have also shown, for comparison, the slope corresponding to a trivial behaviour, $h \sim 1/w$.

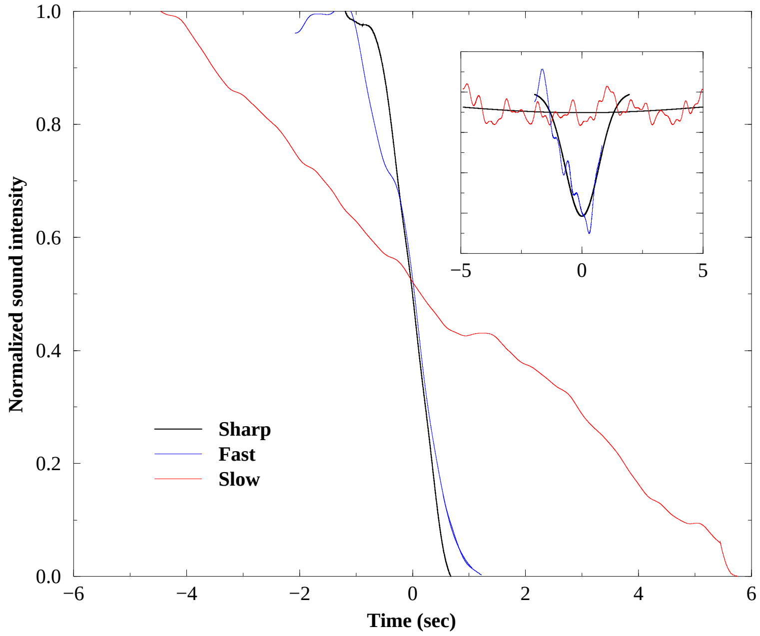

FIG. 18: Terminal stages of clapping corresponding to three characteristic recordings: one of them is a slow decay of applause (over 10 seconds), corresponding to a very heterogeneous audience. The two other ones are fast events (on the order of one second), one of them can even be classified as instantaneous since its width cannot be resolved (i.e. it is thinner than the Sabine reverberation time of the room (≈ 1.8 sec).